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Analytical Approach of Tangency on Geometry Problem

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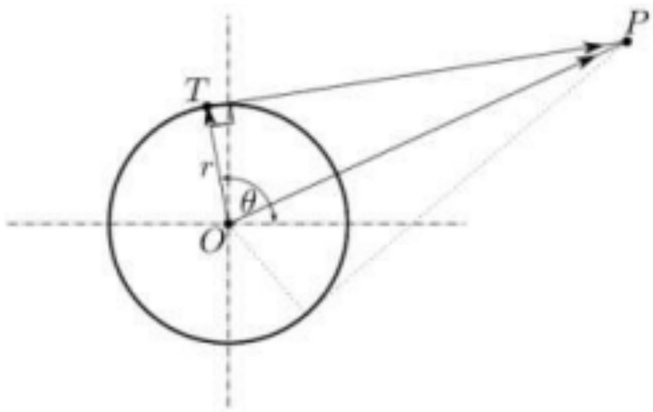

Figure 1. Schematic diagram of the problem[1]

Analytical solution:

1st Approach:

From your final tangent line equation, it gives the following result:

$$(x_P - x_O)cos\theta + (y_P - y_O)sin\theta - r = 0$$

In behalf of algebraic manipulation, let define Δx and Δy as given below:

$$\Delta x \equiv x_p - x_O \; ; \; \Delta y \equiv y_p - y_O$$

then,

$\Delta x cos\theta + \Delta y sin\theta - r = 0$	(1.1)

Let's recall trigonometry identity as given below:

$$sin^2\theta + cos^2\theta = 1 \leftrightarrow \; cos^2\theta = 1 - sin^2\theta$$

Next, let's rearrange the form of (1.1) by taking square on both of sides as follows:

$(r - \Delta y sin\theta)^2 = (\Delta x cos\theta)^2 \; \leftrightarrow \; r^2 - 2r\Delta y sin\theta + \Delta y^2 sin^2\theta = \Delta x^2 cos^2\theta$	(1.2)

Now, let's substitute $cos^2\theta$ from trigonometry identity into (1.2) and by a little bit rearrangement of (1.2), it gives

$sin^2\theta - \frac{2r\Delta y}{\Delta x^2 + \Delta y^2} sin\theta + \frac{r^2 - \Delta x^2}{\Delta x^2 + \Delta y^2} = 0$	(1.3)

It can be viewed that (1.3) resembles the general quadratic equation in which $sin\theta$ as an unknown variable. Otherwise, I can apply "**abc-formula**" expressed below:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \; ; D = b^2 - 4ac$$

in this case, $a = 1; b = \frac{-2r\Delta y}{\Delta x^2 + \Delta y^2}; c = \frac{r^2 - \Delta x^2}{\Delta x^2 + \Delta y^2}$

then,

$$sin\theta_{1,2} = \frac{-\left(\frac{-2r\Delta y}{\Delta x^2 + \Delta y^2}\right) \pm \sqrt{\left(-\frac{2r\Delta y}{\Delta x^2 + \Delta y^2}\right)^2 - 4(1)\left(\frac{r^2 - \Delta x^2}{\Delta x^2 + \Delta y^2}\right)}}{2(1)}$$

by a little bit simplification, it can be obtained that

$$sin\theta_{1,2} = \frac{r\Delta y \pm \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}$$	(1.4)

From (1.4), It has two distinct solutions which are θ_1 and θ_2. Therefore, both of θ_1 and θ_2 are expressed below:

$$\theta_1 = \sin^{-1}\left(\frac{r\Delta y + \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)$$ $$\theta_2 = \sin^{-1}\left(\frac{r\Delta y - \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)$$	(1.5)

It's known that both of θ_1 and θ_2 expressed in (1.5) are evaluated in radian unit (not in degree unit).

Moreover, the coordinate of tangency point (X_T, Y_T) associated with θ_1 and θ_2 is given below:

$$X_{T,1} = X_O + rcos\theta_1;\ Y_{T,1} = Y_O + rsin\theta_1$$
$$X_{T,2} = X_O + rcos\theta_2;\ Y_{T,2} = Y_O + rsin\theta_2$$

Hence,

The coordinate of 1st tangency point $(X_{T,1}, Y_{T,1})$:

$$X_{T,1} = X_O + rcos\theta_1 = X_O + rcos\left(\sin^{-1}\left(\frac{r\Delta y + \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)\right)$$

$$Y_{T,1} = Y_O + rsin\theta_1 = Y_O + rsin\left(\sin^{-1}\left(\frac{r\Delta y + \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)\right)$$

The coordinate of 2nd tangency point $(X_{T,2}, Y_{T,2})$:

$$X_{T,2} = X_O + rcos\theta_2 = X_O + rcos\left(\sin^{-1}\left(\frac{r\Delta y - \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)\right)$$

$$Y_{T,2} = Y_O + rsin\theta_2 = Y_O + rsin\left(\sin^{-1}\left(\frac{r\Delta y - \Delta x\sqrt{\Delta x^2 + \Delta y^2 - r^2}}{\Delta x^2 + \Delta y^2}\right)\right)$$

2nd Approach:

Let introduce a new variable q in which q defined as the following:

$$q \equiv \tan\left(\frac{\theta}{2}\right)$$

The value of $sin\theta$ and $cos\theta$ represented in term of q given as the following:

$$sin\theta = 2sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = 2\left(\frac{q}{\sqrt{1+q^2}}\right)\left(\frac{1}{\sqrt{1+q^2}}\right) = \frac{2q}{1+q^2}$$

$$cos\theta = 2cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\frac{1}{\sqrt{1+q^2}}\right)^2 - 1 = \frac{1-q^2}{1+q^2}$$

Then, substitute these $sin\theta$ and $cos\theta$ into (1), it gives

$\Delta x\left(\frac{1-q^2}{1+q^2}\right) + \Delta y\left(\frac{2q}{1+q^2}\right) - r = 0 \leftrightarrow \Delta x(1-q^2) + 2\Delta y q - r(1+q^2) = 0$	(2.1)

Now, let's rearrange and simplify (2.1) into the general quadratic equation, it follows that

$\Delta x(1-q^2) + 2\Delta y q - r(1+q^2) = 0 \leftrightarrow q^2 - \frac{2\Delta y}{r+\Delta x}q + \frac{r-\Delta x}{r+\Delta x} = 0$	(2.2)

Let's apply abc-formula and in this case $a = 1; b = -\frac{2\Delta y}{r+\Delta x}; c = \frac{r-\Delta x}{r+\Delta x}$

$q_{(1,2)} = \frac{\frac{2\Delta y}{r+\Delta x} \pm \sqrt{\left(\frac{2\Delta y}{r+\Delta x}\right)^2 - 4(1)\left(\frac{r-\Delta x}{r+\Delta x}\right)}}{2} = \frac{\Delta y \pm \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r+\Delta x}$ $q_{(1,2)} = tan\left(\frac{\theta}{2}\right)_{1,2} = \frac{\Delta y \pm \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r+\Delta x}$	(2.3)

From (2.3), It has two distinct solutions which are θ_1 and θ_2. Therefore, both of θ_1 and θ_2 are expressed below:

$\theta_1 = 2\tan^{-1}\left(\frac{\Delta y + \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r+\Delta x}\right)$ $\theta_2 = 2\tan^{-1}\left(\frac{\Delta y - \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r+\Delta x}\right)$	(2.4)

It's known that both of θ_1 and θ_2 expressed in (2.4) are evaluated in radian unit (not in degree unit).

Moreover, the coordinate of tangency point (X_T, Y_T) associated with θ_1 and θ_2 is given below:

$$X_{T,1} = X_O + rcos\theta_1;\ Y_{T,1} = Y_O + rsin\theta_1$$
$$X_{T,2} = X_O + rcos\theta_2;\ Y_{T,2} = Y_O + rsin\theta_2$$

Hence,

The coordinate of 1st tangency point $(X_{T,1}, Y_{T,1})$:

$$X_{T,1} = X_O + r cos\theta_1 = X_O + r cos\left(2\tan^{-1}\left(\frac{\Delta y + \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r + \Delta x}\right)\right)$$

$$Y_{T,1} = Y_O + r sin\theta_1 = Y_O + r sin\left(2\tan^{-1}\left(\frac{\Delta y + \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r + \Delta x}\right)\right)$$

The coordinate of 2nd tangency point $(X_{T,2}, Y_{T,2})$:

$$X_{T,2} = X_O + r cos\theta_2 = X_O + r cos\left(2\tan^{-1}\left(\frac{\Delta y - \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r + \Delta x}\right)\right)$$

$$Y_{T,2} = Y_O + r sin\theta_2 = Y_O + r sin\left(2\tan^{-1}\left(\frac{\Delta y - \sqrt{\Delta x^2 + \Delta y^2 - r^2}}{r + \Delta x}\right)\right)$$

Reference:

[1]. https://spbhat.in/blogs/tangent/index.html, accessed on Friday, July 5, 2019 at 4.14 PM